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PICARD GROUPS AND INFINITE MATRIX RINGS

GENE ABRAMS AND JEREMY HAEFNER

Abstract. We describe a connection between the Picard group of a ring with local units T and the Picard group of the unital overring $End({}_T T)$. Using this connection, we show that the three groups $Pic(R)$, $Pic(FM(R))$, and $Pic(RFM(R))$ are isomorphic for any unital ring R. Furthermore, each element of $Pic(RFM(R))$ arises from an automorphism of $RFM(R)$, which yields an isomorphsm between $Pic(RFM(R))$ and $Out(RFM(R))$. As one application we extend a classical result of Rosenberg and Zelinsky by showing that the group $Out_R(RFM(R))$ is abelian for any commutative unital ring R.

We recall (see e.g. [3]) that a ring A is said to have *local units* in case A contains a set of idempotents E for which $A = \bigcup_{e \in E} eAe$. In particular, any unital ring has local units. In this article we describe various connections between the Picard group of a ring with local units and the Picard groups of related unital rings. The motivation for this investigation stems from some work of Beattie and del Río [4], which includes an investigation of the Picard group of the category of graded modules over a ring graded by an infinite group. They show that a generalization of the standard exact sequence $1 \to Inn(A) \to Aut(A) \to Pic(A)$ arises, where one must consider automorphisms, inner automorphisms, and invertible bimodules in the more general local units context.

For any ring with local units A the ring $B = End({}_A A)$ is a unital ring into which A embeds (via right multiplications) as a right ideal. (If A itself is unital then $A = B$.) Our goal here is to investigate the relationship between the aforementioned exact sequence and the exact sequence $1 \to Inn(B) \to Aut(B) \to Pic(B)$.

In Section 1 we show that there is an isomorphism between a subgroup J of $Pic(A)$ and a subgroup H of $Pic(B)$. In Section 2 we apply this isomorphism in the specific case where $A = FM(R)$ (the ring of countably infinite matrices with entries from the unital ring R, whose elements have almost all components equal to 0). In this situation we get $B = RFM(R)$, the (unital) ring of row-finite matrices over R. We show that $J = Pic(FM(R))$ and $H = Pic(RFM(R))$, and each in turn is isomorphic to $Pic(R)$. We discuss how this result can be interpreted in the context of recent work by Guralnick and Montgomery [11]. We then apply this isomorphism to conclude that the group $Out_R(RFM(R))$ of outer automorphisms of $RFM(R)$ which fix R elementwise is abelian whenever R is a commutative unital ring. This extends the well-known result of Rosenberg and Zelinsky [12, Corollary 6] to the infinite matrix setting.

Throughout this paper A will denote a ring with local units with set of idempotents E; B will denote $End({}_A A)$. For each $a \in A$ we have $\rho_a \in B$ via $(x)\rho_a = xa$;

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the map $\rho : A \to B$ via $a \mapsto \rho_a$ gives an embedding of A in B as a right ideal. We will often identify $a \in A$ with its image $\rho_a \in B$ without explicit mention.

A left A-module M is called *unitary* in case $AM = M$; the category A-*mod* is defined to be the collection of unitary left A-modules, together with usual homomorphisms. Unless otherwise indicated, the word *module* (resp. *bimodule*) will always mean *unitary module* (resp. *unitary bimodule*). All module homomorphisms will be written opposite the scalars.

Analogous to the definition for unital rings, the group $Pic(A)$ consists of those $A-A$ bimodules P which are both left and right unitary, and for which there exists a (left and right unitary) $A-A$ bimodule Q having $P \otimes Q \cong A$ and $Q \otimes P \cong A$ as bimodules. The operation in $Pic(A)$ is $\otimes_A$. By [3, Theorem 2.2] $Pic(A)$ is precisely the group of category autoequivalences of A-*mod*.

If ϕ is an automorphism of the ring T and X is any subset of T, we often denote the set $(X)\phi$ by X^ϕ. For $\phi \in Aut(T)$ we construct the $T-T$ bimodule T_ϕ which as a set is T, has the same left module structure as ${}_TT$, and as a right T-module has structure given by setting $t * x = tx^\phi$ for $t, x \in T$. It is straightforward to show that even in the more general setting of rings with local units we have $T_\phi \in Pic(T)$ (with inverse $T_{\phi^{-1}}$), and the map $\phi \mapsto T_\phi$ is a group homomorphism from $Aut(T)$ to $Pic(T)$. If T is unital, we say that $Pic(T)$ is *outer induced* in case this homomorphism is surjective. (Bolla [5] calls such rings *rings having the Aut-Pic property.*) The kernel of this homomorphism in the unital ring situation is the subgroup $Inn(T)$ of inner automorphisms of T; thus there is an embedding of the group $Out(T) = Aut(T)/Inn(T)$ into $Pic(T)$, which is an isomorphism in case $Pic(T)$ is outer induced. More generally, if A is a ring with local units, Beattie and del Río define $Inn(A)$ to be the kernel of the homomorphism $\phi \mapsto T_\phi$. In this situation the elements of $Inn(A)$ are explicitly described in [4, Section 1].

Following [10], if T is any ring and M is a left T-module then $Div(M)$ denotes those left T-modules N such that N is isomorphic to a direct summand of a finite direct sum of copies of M; i.e., those N for which there is a split epimorphism $M^s \to N$ for some integer s.

1. An isomorphism between subgroups of $Pic(A)$ and $Pic(B)$

In this section we show that there is a strong connection between the Picard groups $Pic(A)$ and $Pic(B)$. Specifically, we show that for any ring with local units A there are subgroups J of $Pic(A)$ and H of $Pic(B)$ for which $J \cong H$. We then show (in Section 2 as well as in [1]) that in many situations we have $J = Pic(A)$ and $H = Pic(B)$, thus yielding an isomorphism between the two relevant Picard groups.

For any right A-module M_A we define a right B-module structure on M by setting $m * b = m(eb)$, where $e \in E$ has $me = m$. It is easy to show that this is well-defined. In fact, M_B is isomorphic to the tensor product $M \otimes_A A_B$, via the map $m \otimes a \mapsto ma$ (whose inverse is given by the map $m \mapsto m \otimes e$ where $e \in E$ has $me = m$). In case $M = A_A$ this is just right multiplication of B on A. Given this, for any $A-A$ bimodule ${}_AX_A$ we can define a $B-B$ bimodule structure on

$$\widehat{X} = Hom_A(A, X)$$

by viewing $\widehat{X}$ as $Hom_A({}_AA_B, {}_AX_B)$. Specifically, for $b \in B$ and $f \in Hom_A(A, X)$ we define $b * f = \rho_b \circ f$, where ρ_b denotes right multiplication by b. Similarly, we define $f * b$ by setting (for each $a \in A$) $(a)f * b = (a)f \cdot eb$, where $e \in E$ has $(a)f \cdot e = (a)f$. As abelian groups we always have X embedded in $\widehat{X}$ by $x \mapsto \rho_x$; in general this embedding is not surjective. Of course, if A is unital, then $A = B$ and ρ_x is an isomorphism for all $x \in X$.

Our immediate goal is to find conditions which ensure that $\widehat{X} \in Pic(B)$ for $X \in Pic(A)$. We start with some preliminary results regarding the elements of the Picard group of a ring with local units.

Notation 1.1. (1) If X is an $A - A$ bimodule, then there is a ring homomorphism $\rho^X : B \to End({}_AX)$ given by $b \mapsto \rho_b^X$, where $(x)\rho_b^X = x * b$.

(2) If $X \in Pic(A)$, then by definition there exists an $A-A$ bimodule isomorphism $X^{-1} \otimes X \cong A$; we denote this by $\xi^X : X^{-1} \otimes X \to A$.

Proposition 1.2. *Suppose $X \in Pic(A)$. Then $End({}_AX) \cong B$ both as rings and as $B - B$ bimodules, via the map $\rho^X : B \to End({}_AX)$ which takes b to ρ_b^X.*

Proof. Let $C = End({}_AX)$. It is clear that ρ^X is a ring map, since $(bb')\rho^X = \rho^X_{bb'} = \rho_b^X \cdot \rho_{b'}^X$. We define $\theta^X : C \to B$ by setting $(g)\theta^X = (\xi^X)^{-1}(1_{X^{-1}} \otimes g)\xi^X$. Now θ^X is a ring map, as for $g, g' \in C$ we have

$$(gg')\theta^X = (\xi^X)^{-1}(1_{X^{-1}} \otimes gg')\xi^X = (\xi^X)^{-1}(1_{X^{-1}} \otimes g) \cdot (1_{X^{-1}} \otimes g')\xi^X$$

$$= (\xi^X)^{-1}(1_{X^{-1}} \otimes g)\xi^X \cdot (\xi^X)^{-1}(1_{X^{-1}} \otimes g')\xi^X = (g)\theta^X \cdot (g')\theta^X.$$

We now show that $\rho^X\theta^X = 1_B$, which in particular means that ρ^X is injective and θ^X is surjective. Let $b \in B$. Then $(b)\rho^X\theta^X = (\rho_b^X)\theta^X = (\xi^X)^{-1}(1_{X^{-1}} \otimes \rho_b^X)\xi^X$. We show that for $a \in A$, $(a)(\xi^X)^{-1}(1_{X^{-1}} \otimes \rho_b^X)\xi^X = ab$. But writing $(a)(\xi^X)^{-1} = \sum x_i' \otimes x_i$, where $x_i' \in X^{-1}$ and $x_i \in X$, we have

$$(a)(\xi^X)^{-1}(1_{X^{-1}} \otimes \rho_b^X)\xi^X = (\sum x_i' \otimes x_i)(1 \otimes \rho_b^X)\xi^X = (\sum x_i' \otimes x_i b)\xi^X.$$

Since the sum is finite, there exists $e = e^2 \in A$ such that $ae = a$ and $x_i e = x_i$ for each i. Thus,

$$(a)(\xi^X)^{-1}(1_{X^{-1}} \otimes \rho_b^X)\xi^X = (\sum x_i' \otimes x_i eb)\xi^X = \sum (x_i' \otimes x_i eb)\xi^X$$

$$= \sum (x_i' \otimes x_i)\xi^X(eb) = (\sum x_i' \otimes x_i)\xi^X(eb) = a(eb) = ab,$$

where the eb may be factored out since ξ^X is a right A-map. Thus, for $a \in A$, $(a)(\xi^X)^{-1}(1_{X^{-1}} \otimes \rho_b^X)\xi^X = ab$, and so

$$(b)\rho^X\theta^X = (\rho_b^X)\theta^X = (\xi^X)^{-1}(1_{X^{-1}} \otimes \rho_b^X)\xi^X = \rho_b^A.$$

Since the action of B on A is given by right multiplication by the maps ρ_b^A, we see that $\rho_b^A = b$, and so the claim is shown.

Now suppose $c \in C$ is such that $(c)\theta^X = 0$. Then $(\xi^X)^{-1}(1 \otimes c)\xi^X = 0$ in B. It follows that $1 \otimes c = 0$. But since X is invertible, we see that $c : X \to X$ is the zero map and so θ^X is injective.

Thus we have that θ^X is a ring isomorphism. To see that ρ^X is the inverse of θ^X, notice that

$$\rho^X = \rho^X \cdot 1_C = \rho^X(\theta^X[\theta^X]^{-1}) = 1_B[\theta^X]^{-1} = [\theta^X]^{-1}.$$

Next we prove that ρ and θ are bimodule maps. We define a $B - B$ bimodule structure on $C = End({}_AX)$ as follows: for $c \in C$ and $b, b' \in B$, $b * c * b' := \rho_b^X \cdot c \cdot \rho_{b'}^X$.

Thus, if $\beta \in B$, then $(b\beta b')\rho^X = \rho_b^X \rho_\beta^X \rho_{b'}^X = b*(\beta)\rho^X * b'$, so ρ^X is a $B-B$ bimodule map.

In a similar manner,

$$(b * c * b')\theta^X = (\rho_b^X c \rho_{b'}^X) = (\rho_b^X)\theta^X (c)\theta^X (\rho_{b'}^X)\theta^X = b(c)\theta^X b'. \quad \square$$

Remark 1.3. (1) If $X \in Pic(A)$ and $g \in End({}_AX)$, then

$$(x)g = x * (g)\theta^X,$$

where $*$ denotes the induced action of B on X. To see this, recall that $\theta^X \rho^X = 1_{End({}_AX)}$. Then for each $x \in X$ we have

$$(x)g = (x)g \circ 1_{End({}_AX)} = (x)(g)\theta^X \rho^X (x)\rho^X_{(g)\theta^X} = x * (g)\theta^X.$$

(2) The $A-A$ bimodule isomorphism $\xi^X : X^{-1} \otimes X \to A$ satisfies the property

$$(x' \otimes (x)g)\xi^X = (x' \otimes x)\xi^X (g)\theta^X$$

for each $x' \in X^{-1}$, $x \in X$, and $g \in End({}_AX)$. This is because $(x' \otimes (x)g)\xi^X = (x' \otimes x * (g)\theta^X)\xi^X = (x' \otimes x * e(g)\theta^X)\xi^X = (x' \otimes x)\xi^X e(g)\theta^X = (x' \otimes x)\xi^X (g)\theta^X$.

For a unital ring C the regular module ${}_CC$ is finitely generated; furthermore, if $P \in Pic(C)$ then ${}_CP$ is finitely generated. These two properties together in turn yield that $P \in Div(C)$ and $C \in Div(P)$. For a nonunital ring A with local units the regular module ${}_AA$ is not finitely generated; this implies that an element X of $Pic(A)$ need not be in $Div(A)$, and that A need not be in $Div(X)$; see e.g. Example 1.15 below. We now show that those elements X of $Pic(A)$ for which $X \in Div(A)$ and $A \in Div(X)$ play a key role in our investigation.

Proposition 1.4. *Let $X \in Pic(A)$. Then $\widehat{X} \in Pic(B)$ if and only if $X \in Div(A)$ and $A \in Div(X)$.*

Proof. ($\Rightarrow$) Assume $\widehat{X} \in Pic(B)$. Then we have in particular that ${}_B\widehat{X}$ is finitely generated and projective. So there exists a split epimorphism of left B-modules $\widehat{\alpha} : {}_BB^n \to \widehat{X}$. Denote by $\widehat{\beta}$ a splitting map from $\widehat{X}$ to ${}_BB^n$ for $\widehat{\alpha}$; so for each $f \in \widehat{X}$ we have $(f)\widehat{\beta}\widehat{\alpha} = f$.

For $1 \le i \le n$ let $f_i \in \widehat{X} = Hom_A(A, X)$ denote the image under $\widehat{\alpha}$ of the standard basis element e_i of B^n. Since $\widehat{\alpha}$ is surjective we have that $\{f_1, ..., f_n\}$ generate $\widehat{X}$ as a left B-module. Define $\alpha : A^n \to X$ by setting $\alpha : (a_1, ..., a_n) \mapsto \sum_{i=1}^n (a_i)f_i$ for all $(a_1, ..., a_n) \in A^n$. Then it is easy to show that α is a homomorphism of left A-modules. Now for $x \in X$ define $(x)\beta = (\rho_x)\widehat{\beta}$. Using the fact that $ex = x$ for some $e \in E$, it is easy to show that $(x)\beta \in A^n$. A tedious but straightforward computation yields that β splits α.

We now show that there is a split epimorphism $\gamma : X^m \to A$ in *A-mod*. But $\widehat{X} \in Pic(B)$ gives that ${}_B\widehat{X}$ is a generator of *B-mod*. In particular there exists a split epimorphism $\widehat{\gamma} : \widehat{X}^m \to B \to 0$ in *B-mod*. For $(x_1, ..., x_m) \in X^m$ define $\gamma : X^m \to A$ by setting $(x_1, ..., x_m)\gamma = (\rho_{x_1}, ..., \rho_{x_m})\widehat{\gamma}$. By using methods similar to those above, it is easy to show that γ in fact maps into A, and that γ is left A-linear. For $a \in A$ define $(a)\delta = ((a)f_1, ..., (a)f_m) \in X^m$, where $(f_1, ..., f_m) \in \widehat{X}^m$ has $(f_1, ..., f_m)\widehat{\gamma} = 1_A$. Then δ is easily shown to split γ.

($\Leftarrow$) Suppose there exist a split epimorphism $\alpha : A^n \to X$ and a split epimorphism $X^m \to A \to 0$ in *A-mod*. The goal is to show that $\widehat{X} \in Pic(B)$. We do this by showing that there is a Morita context in which $\widehat{X}$ is one of the bimodules. Define the maps $\tau : \widehat{X} \otimes_B Hom_A(X, A) \to Hom_A(A, A) = B$ and $\psi : Hom_A(X, A) \otimes_B \widehat{X} \to Hom_A(X, X)$ by appropriately extending the composition map in each case. Let μ denote the composition $\psi \circ \theta^X : Hom_A(X, A) \otimes_B \widehat{X} \to B$ where θ^X is described in the proof of Proposition 1.2.

Now recall the $B-B$ bimodule action on $Hom_A(X, A)$ and $Hom_A(A, X)$ respectively: for $f \in Hom_A(X, A)$, $g \in Hom_A(A, X)$ and $b, b' \in B$, we have

$$b * f * b' := \rho_b^X \circ f \circ \rho_{b'}^A \quad \text{and} \quad b * g * b' := \rho_b^A \circ g \circ \rho_{b'}^X.$$

We show that the maps τ and μ satisfy the Morita pair conditions (see e.g. [2, p. 266]); namely, for $h, f \in Hom_A(X, A)$ and $k, g \in Hom_A(A, X)$ we have

$$(1) \quad (f \otimes g)\mu * h = f * (g \otimes h)\tau \qquad \text{and} \qquad (2) \quad k * (f \otimes g)\mu = (k \otimes f)\tau * g.$$

For (1) we have

$$(f \otimes g)\mu * h := \rho^X_{(f\otimes g)\mu} \circ h = \rho^X_{(f\circ g)\theta^X} \circ h = (f \circ g)\theta^X \rho^X \circ h$$

$$= (f \circ g) \circ h = (fg)h = f(gh) = f * (g \otimes h)\tau.$$

For (2) we have:

$$k * (f \otimes g)\mu := k \circ \rho^X_{(f\otimes g)\mu} = k \circ \rho^X_{(fg)\theta^X} = k \circ [fg]\theta^X \rho^X$$

$$= k \circ (fg) = k(fg) = (kf)g = (k \otimes f)\tau * g.$$

Now let T and W be arbitrary rings, and let ${}_TM$ and ${}_TN_W$ be modules. Let $f : Hom_T(M, N_W) \otimes_W Hom_T(N_W, M) \to Hom_T(M, M)$ be the extension of the composition map. Suppose there exists a split epimorphism $\nu : {}_TN^j \to {}_TM$ for some integer j (say with splitting map χ). Then f is an epimorphism. To see this, let $\pi_i : N^j \to N$ be the ith coordinate projection, and let $\iota_i : N \to N^j$ be the ith coordinate inclusion. So $\sum \pi_i \circ \iota_i$ is the identity on N^j. Now let $\alpha \in Hom_T(M, M)$. Then for each i we have that $\alpha \circ \chi \circ \pi_i \in Hom_T(M, N)$, and $\iota_i \circ \nu \in Hom_T(N, M)$. We compute:

$$(\sum_{i=1}^{j} \alpha \circ \chi \circ \pi_i \otimes \iota_i \circ \nu)f = \sum_{i=1}^{j} \alpha \circ \chi \circ \pi_i \circ \iota_i \circ \nu = \alpha \circ \chi \circ (\sum_{i=1}^{j} \pi_i \circ \iota_i) \circ \nu$$

$$= \alpha \circ \chi \circ 1_{N^j} \circ \nu = \alpha \circ \chi \circ \nu = \alpha \circ 1_N = \alpha.$$

For the specific case at hand, the result of the preceding paragraph, together with the hypotheses, yield that both τ and μ are epimorphisms. Therefore the equivalence data $(B, B, \widehat{X}, Hom_A(X, A), \tau, \mu)$ has both maps surjective, hence yields a Morita equivalence (see e.g. [2, Section 22]). Specifically, this means that tensoring by $\widehat{X}$ and by $Hom_A(X, A)$ each induces a category equivalence from B-*mod* to itself; i.e., that $\widehat{X}$ is an element of $Pic(B)$. □

We note that the proof of the above proposition in fact yields

Corollary 1.5. *Suppose $X \in Pic(A)$ has $\widehat{X} \in Pic(B)$. Then $\widehat{X}^{-1} = Hom_A(X, A)$. In particular, $Hom_A(X, A) \in Pic(B)$.*

With Proposition 1.4 in mind, we define

$$J = \{X \in Pic(A) \mid X \in Div(A) \text{ and } A \in Div(X)\}.$$

Of course, if A is unital then $J = Pic(A)$. In general, however, J may indeed be proper in $Pic(A)$; see Example 1.15 below. We now show that the map $X \mapsto \widehat{X}$ is multiplicative; this will consequently give that J is a subgroup of $Pic(A)$. We start by recording some useful properties.

Lemma 1.6. *(1) Let W be any left B-module. Then $AW = \{\sum a_i w_i \mid a_i \in A, w_i \in W\}$ is the largest (unitary) A-submodule of W. Moreover, $A \otimes W \cong AW$ as left A-modules via the map μ where $(a \otimes w)\mu = aw$. Similar statements hold for right B-modules.*

(2) Let V be any left A-module. Then $A \otimes_B Hom_A(A, V) \cong V$ as left A-modules. In addition, if V_C has a right module structure for some ring C, then this isomorphism is also a right C-module isomorphism.

(3) $Hom_A(A, A \otimes_B Hom_A(A, V)) \cong Hom_A(A, V)$ as left B-modules. In addition, if V_C has a right module structure for some ring C, then this isomorphism is also a right C-module isomorphism.

(4) Let M be a $D - B$ bimodule, and let N be a $B - C$ bimodule (where C and D are arbitrary rings). Then $MA \otimes_B N \cong MA \otimes_A AN$ as $D - C$ bimodules. In particular, for any $D - A$ bimodule M, $M \otimes_B N \cong M \otimes_A AN$ as $D - C$ bimodules.

Proof. (1) The first statement is clear. Define $\zeta : AW \to A \otimes_B W$ by setting $(\sum a_i w_i)\zeta = e \otimes \sum a_i w_i$, where $e \in E$ has $e \sum a_i w_i = \sum a_i w_i$. It is easy to show that ζ is well defined, a left A-homomorphism, and the inverse of μ.

(2) $A \otimes_B Hom_A(A, V) \cong AHom_A(A, V)$ by part (1). But $AHom_A(A, V) \cong V$ as left A-modules by [3, Proposition 1.1]. The isomorphism clearly preserves right structures.

(3) That these are isomorphic as abelian groups follows from (2). But the induced map from $Hom_A(A, A \otimes_B Hom_A(A, V))$ to $Hom_A(A, V)$ clearly preserves the left B-action, as well as any right action on V.

(4) It is easy to show that the map $m \otimes_B n \mapsto m \otimes_A n$ is an isomorphism of the desired type. □

Lemma 1.7. *Let P be any finitely generated projective left B-module (i.e. $P \in Div(B)$).*

(1) $\widehat{AP} = Hom_A(A, A \otimes_B P)$ is isomorphic to P as left B-modules. If P_C is also a right C-module for some ring C, then this isomorphism is as $B - C$ bimodules.

(2) If ${}_AU \in Div(A)$, then the homomorphism

$$\eta : Hom_A(U, A) \otimes_B P \to Hom_A(U, A \otimes_B P)$$

given by setting $(u)[(\gamma \otimes p)\eta] = (u)\gamma \otimes p$ for each $u \in U$ is an isomorphism of abelian groups. This isomorphism is a $D - C$ bimodule isomorphism whenever we have right module structures U_D and P_C for rings C and D.

(3) If ${}_AX_B$ is a bimodule (e.g. if $X \in Pic(A)$ and X is given the right B-structure described above), then the homomorphism $\eta : Hom_A(A, X) \otimes_B P \to Hom_A(A, X \otimes_B P)$ given by setting $(a)[(\gamma \otimes p)\eta] = (u)\gamma \otimes p$ for each $a \in A$ is an isomorphism of abelian groups. This isomorphism is a $D - C$ bimodule isomorphism whenever we have right module structures A_D and P_C for rings C and D.

Proof. For each of the three statements we will invoke the idea described in [2, Proposition 20.10]. Specifically, if a natural transformation ζ has the property that ζ_M is an isomorphism for some module M, then ζ_N is an isomorphism for any $N \in Div(M)$.

(1) Let F denote the functor $F : B\text{-}mod \to B\text{-}mod$ given by

$$F(M) = Hom_A(A_B, A \otimes_B M),$$

and let $I : B\text{-}mod \to B\text{-}mod$ denote the identity functor. For each $m \in M$ let $\overline{\rho}_m : A \to A \otimes_B M$ denote the map given by $(a)\overline{\rho}_m = a \otimes m$; then $\overline{\rho}_m \in F(M)$. Now let $\xi : I \to F$ denote the transformation given by setting, for each $M \in B\text{-}mod$ and each $m \in M$, $(m)\xi_M = \overline{\rho}_m$. It is easy to check that ξ is a natural transformation.

We claim that $\xi_B : B \to Hom_A(A_B, A \otimes_B B)$ is an isomorphism of abelian groups. But this is clear: if $b \in B$ has $a \otimes b = 0$ for all $a \in A$ then $b = 0$ (since $A \otimes B \cong A$ as $A - B$ bimodules, and the annihilator in B of A is zero). Similarly, any A-homomorphism from A to $A \otimes B$ is just right multiplication by some element of B, since $A \otimes B \cong AB \cong A$ as left A-modules by Lemma 1.6(1).

But as ξ is a natural transformation we can use the fact that ξ_B is an isomorphism together with the aforementioned property to conclude that ξ_P is an isomorphism of abelian groups whenever $P \in Div(B)$. The computation $(a)(bp)\xi_P = (a)\overline{\rho}_{bp} = a \otimes bp = ab \otimes p = (ab)\overline{\rho}_p = (a)\rho_b \circ \overline{\rho}_p = (a)b * \overline{\rho}_p = (a)b * (p)\xi_P$ (for each $a \in A, b \in B, p \in P$) yields that ξ_P is in fact a left B-module isomorphism. It is clear that the map ξ_P also preserves whatever right structure there might be on P.

(2) We first consider the particular case where $U = A$ as left A-modules. Then this homomorphism is the map $\eta_A : Hom_A(A, A) \otimes_B P \to Hom_A(A, A \otimes_B P)$. The domain of η_A is isomorphic to $B \otimes_B P \cong P$, and the codomain is also isomorphic to P by part (1). It can be shown that the map η_A is the composition of these various isomorphisms. Thus $\eta_U : Hom_A(U, A) \otimes_B P \to Hom_A(U, A \otimes_B P)$ is an isomorphism for any $U \in Div(A)$. By tracing through the appropriate maps, it is easy to show that in fact this is a left D-module map whenever we have U_D, and a right C-module map whenever we have P_C.

(3) The homomorphism $\eta_B : Hom_A(A, X) \otimes_B B \to Hom_A(A, X \otimes_B B)$ analogous to that described in part (2) is an isomorphism; in fact, each of the appropriate modules is isomorphic to $Hom_A(A, X)$. The result now follows as above. □

Proposition 1.8. *Let $X, Y \in J$. Then as $B - B$ bimodules we have*

(1) $\widehat{X \otimes_A Y} \cong \widehat{X} \otimes_B \widehat{Y}$, and

(2) $\widehat{X^{-1}} \cong \widehat{X}^{-1}$.

Proof. (1) As $Y \in J$, we have by Proposition 1.4 that $\widehat{Y} = Hom_A(A, Y)$ is in $Pic(B)$, so ${}_B\widehat{Y}$ is finitely generated projective. Thus Lemma 1.7(3) applies to give the isomorphism

$$\eta : \widehat{X} \otimes_B \widehat{Y} = Hom_A(A, X) \otimes_B Hom_A(A, Y) \\ \to Hom_A(A, X \otimes_B Hom_A(A, Y)).$$

But ${}_AX_B = {}_AX_A \otimes A_B$ as $A - B$ bimodules by the remark made at the beginning of this section, so $X \otimes_B Hom_A(A, Y) \cong X_A \otimes A_B \otimes_B Hom_A(A, Y) \cong X_A \otimes Y$ (by Lemma 1.6(2)). Thus we have an isomorphism

$$\kappa : Hom_A(A, X \otimes_B Hom_A(A, Y)) \to Hom_A(A, X \otimes_A Y) = \widehat{X \otimes_A Y}.$$

The composition $\eta\kappa$ provides the desired isomorphism, as it is easily checked that $\eta\kappa$ preserves both the left and right B-structures of the appropriate modules.

(2) Let Y denote X^{-1} in $Pic(A)$; so $X \otimes Y \cong A$ and $Y \otimes X \cong A$. As $X \in J$ we have split epimorphisms of left modules $X^n \to A$ and $A^t \to X$ for some integers n, t. Since tensoring preserves split epimorphisms, tensoring each of these maps by Y on the left yields split epimorphisms $Y \otimes X^n \to Y \otimes A$ and $Y \otimes A^t \to Y \otimes X$, which by the standard isomorphisms involving tensor products and direct sums yields split epimorphisms $A^n \to Y$ and $Y^t \to A$. Thus $X^{-1} \in J$. So by Proposition 1.4 we have $\widehat{X^{-1}} = Hom_A(A, X^{-1}) \in Pic(B)$. So we may apply Lemma 1.7(3) above to get an isomorphism

$$Hom_A(A, X) \otimes_B Hom_A(A, X^{-1}) \to Hom_A(A, X \otimes_B Hom_A(A, X^{-1})).$$

But $X \otimes_B Hom_A(A, X^{-1}) \cong A$ as $A - B$ bimodules, since each of these expressions yields X^{-1} upon tensoring both sides on the left by the invertible $A - A$ bimodule X^{-1}. Thus we have an isomorphism $Hom_A(A, X \otimes_B Hom_A(A, X^{-1})) \to Hom_A(A, A) = B$, which is easily shown to be a $B - B$ bimodule isomomorphsm. The composition of these two isomorphisms shows that

$$Hom_A(A, X) \otimes_B Hom_A(A, X^{-1}) \cong B.$$

A virtually identical argument yields that $Hom_A(A, X^{-1}) \otimes_B Hom_A(A, X) \cong B$ as well. Thus we have shown that $\widehat{X^{-1}} \cong \widehat{X}^{-1}$, as required. □

Proposition 1.9. *J is a subgroup of $Pic(A)$, and the map $X \mapsto \widehat{X}$ is a group homomorphism from J to $Pic(B)$.*

Proof. Let $X, Y \in J$. Then $\widehat{X}$ and $\widehat{Y}$ are in $Pic(B)$ by Proposition 1.4, so $\widehat{X} \otimes \widehat{Y} \in Pic(B)$. With Proposition 1.8(1) this yields that $\widehat{X \otimes Y} \in Pic(B)$, so that $X \otimes Y \in J$ (again by Proposition 1.4). Thus J is closed under products. Clearly $A \in J$. Finally, it was shown in the proof of Proposition 1.8(2) that J is closed under inverses.

The second statement follows from Proposition 1.8(1). □

We now identify the corresponding subgroup of $Pic(B)$.

Definition 1.10. Let A be a ring with local units, and let $B = End(_A A)$. We define

$$\begin{aligned} H &= \{P \in Pic(B) \mid AP \text{ and } AP^{-1} \text{ are unitary right } A\text{-modules}\} \\ &= \{P \in Pic(B) \mid AP = APA \text{ and } AP^{-1} = AP^{-1}A\}. \end{aligned}$$

Analogous to the situation relating J to $Pic(A)$, we of course have that $H = Pic(B)$ whenever A is unital (since then $A = B$), but that H may be proper in $Pic(B)$ more generally (see Example 1.18 below).

Proposition 1.11. *H is a subgroup of $Pic(B)$, and the map $P \mapsto AP$ is a group homomorphism from H to $Pic(A)$.*

Proof. Clearly $B \in H$ (as $AB = A$ is right unitary), and H is closed under inverses by definition. If $P, Q \in H$ then we want to show that $A(P \otimes_B Q)$ is right A-unitary. But $A(P \otimes_B Q) = AP \otimes_B Q = APA \otimes_B Q \cong APA \otimes_A AQ$ (by Lemma 1.6(4)) $= APA \otimes_A AQA$. Since the isomorphism is as $A - B$ bimodules, and the last module is right A-unitary, so is $A(P \otimes_B Q)$. The second statement follows directly from this same computation; namely, that $A(P \otimes_B Q) \cong APA \otimes_A AQ = AP \otimes_A AQ$. □

Proposition 1.12. *Let $P \in H$. Then $AP \in J$.*

Proof. We first show that $AP \in Pic(A)$. But AP and AP^{-1} are each unitary (left and) right A-modules by hypothesis; that is, $AP = APA$ and $AP^{-1} = AP^{-1}A$. Now $AP \otimes_A AP^{-1} = APA \otimes_A AP^{-1} = APA \otimes_B P^{-1}$ (by Lemma 1.6(4)) $= AP \otimes_B P^{-1} \cong AB = A$. Similarly one can show that $AP^{-1} \otimes_A AP \cong A$. Thus $AP \in Pic(A)$; in fact, $AP^{-1} \in Pic(A)$, and $(AP)^{-1} = AP^{-1}$. We now must show that $AP \in Div(A)$ and $A \in Div(AP)$. But as B is unital, we have split epimorphisms of left B-modules $B^n \to P$ and $P^t \to B$ for some integers n, t. On tensoring each of these on the left by A_B and using the standard isomorphisms we get the desired properties. □

Proposition 1.13. *Let $X \in J$. Then $\widehat{X} \in H$.*

Proof. By Proposition 1.4 we have that $\widehat{X} \in Pic(B)$. It suffices then to show that $A\widehat{X}$ and $A\widehat{X}^{-1}$ are right A-unitary. By Lemma 1.6(2) we have $A\widehat{X} \cong X$ as $A-A$ bimodules, so that $A\widehat{X}$ is right A-unitary. By Proposition 1.8(2) we have that $\widehat{X^{-1}} \cong \widehat{X}^{-1}$; so $A\widehat{X}^{-1} \cong A\widehat{X^{-1}} \cong X^{-1}$ (again by Lemma 1.6(2)), whence $A\widehat{X}^{-1}$ is also right A-unitary. Thus $\widehat{X} \in H$. □

Finally we are in position to prove the main result of this section.

Theorem 1.14. *Let A be a ring with local units, and let $B = End({}_AA)$. Then $J \cong H$. In particular, the maps $X \mapsto \widehat{X}$ and $P \mapsto AP$ give the appropriate inverse group isomorphisms.*

Proof. With the results of the previous four propositions in mind, the second statement is all that remains to be verified. But for $X \in J$ we have $A\widehat{X} \cong X$ by Lemma 1.6(2). Furthermore, for $P \in H$ we have $\widehat{AP} \cong P$ by Lemma 1.7(1). This completes the proof of the theorem. □

We now give an example of a ring with local units A for which $J \neq Pic(A)$.

Example 1.15. For each positive integer i let R_i denote a unital ring for which there exists an element P_i of $Pic(R)$ with the property that P_i is generated by no less than i elements as a left R_i-module. For instance, let k be a field, let $R_i = M_i(k) \oplus k$, and let P_i be the natural k-complement of R_i when viewed inside $M_{i+1}(k)$. (For a specific description of the case $i = 2$ see Example 2.12 below.) As another example, R_i can be taken to be an integral domain as constructed in [9]. Let A denote the ring direct sum $\bigoplus_{i\in\mathbf{N}} R_i$. Then A is a ring with local units. Moreover, $P = \bigoplus_{i\in\mathbf{N}} P_i$ is an $A-A$ bimodule coordinatewise. Using the fact that tensor products distribute over direct sums, it is easy to see that $P \in Pic(A)$, as $P^{-1} = \bigoplus_{i\in\mathbf{N}} {P_i}^{-1}$. But $P \notin Div(A)$, as no finite direct sum of copies of A can generate all of P. Thus $P \notin J$. (In fact, in the matrix example of the R_i we may also conclude that $A \notin Div(P)$ as well.)

In Example 1.18 below we present a situation in which $H \neq Pic(B)$. To justify this example we first need some properties of automorphisms of B; recall the notation pertaining to these given in the introduction.

Lemma 1.16. *Let A be a ring with local units, and let $B = End({}_AA)$. Let $\sigma \in Aut(B)$, so that $B_\sigma \in Pic(B)$. Then AB_σ is right A-unitary if and only if $A \subseteq B \cdot A^\sigma$.*

Proof. ($\Leftarrow$) As sets we have $AB_\sigma = A$, and the right A-action is given by setting $a * x = ax^\sigma$. So it suffices to show that for $a \in A$ there exists $g \in A$ with $ag^\sigma = a$. There exists $e \in E$ with $ae = a$. The hypotheses yield that there exists $f \in BA$ with $f^\sigma = e$. So $af^\sigma = a$. But $f \in BA$ implies there is $g \in A$ with $fg = f$. Then $ag^\sigma = (af^\sigma)g^\sigma = a(fg)^\sigma = af^\sigma = a$.

($\Rightarrow$) Suppose for each $a \in A$ there exists $e \in A$ with $a = a * e = ae^\sigma$. But then $a = ae^\sigma \in B \cdot A^\sigma$. □

Since $(B_\sigma)^{-1} = B_{\sigma^{-1}}$ in $Pic(B)$, we get, as an immediate consequence,

Corollary 1.17. *Let A be a ring with local units, let $B = End({}_AA)$, and let $\sigma \in Aut(B)$. Then the element B_σ of $Pic(B)$ is in H if and only if $A \subseteq B \cdot A^\sigma$ and $A \subseteq B \cdot A^{\sigma^{-1}}$.*

The following example was suggested by Juan Jacobo Simón.

Example 1.18. Let k be a field, let $T = FM(k)$, and let $R = RFM(k)$. Let $A = R \times T$, the ring direct product of R and T. Then we easily get $B = End({}_AA) = R \times R$. Let $\phi \in Aut(B)$ be the transpose automorphism, given by $(x, y)\phi = (y, x)$. Then $A^\phi = T \times R$, so that $B \cdot A^\phi = RT \times R$. But $A = R \times T$ is not contained in $B \cdot A^\phi = RT \times R$. We now apply Corollary 1.17 to conclude that the element B_ϕ of $Pic(B)$ is not in H.

We take up in [1] the issue of providing sufficient conditions on A and/or B which imply that $J = Pic(A)$ and/or $H = Pic(B)$. In addition, we provide examples in which the Picard groups $Pic(A)$ and $Pic(B)$ are not isomorphic.

2. $Pic(R)$ is isomorphic to $Pic(RFM(R))$

The main result of this second section is that, for any unital ring R, the groups $Pic(R)$ and $Pic(RFM(R))$ are isomorphic. In particular, we conclude that for any unital ring R there is a unital ring B with $Pic(B)$ outer induced for which $Pic(R) \cong Pic(B)$. We will apply this isomorphism to prove an extension of [12, Corollary 6] to infinite matrix rings; specifically, we prove that $Out_R(RFM(R))$ is an abelian group for any unital commutative ring R.

We start by providing a general context for these results. Let R be any unital ring and n any positive integer. Following [11], we define the subgroup I_n of $Pic(R)$ by setting $I_n = \{X \in Pic(R) \mid X^n \cong R^n \text{ as left } R\text{-modules}\}$. By [8, Theorem 55.12], I_1 is the image of $Out(R)$ in $Pic(R)$. More generally, in fact, there is an isomorphism of groups $\alpha_n : Out(M_n(R)) \to I_n$. In addition, there is an embedding $Out(M_n(R)) \subseteq Out(RFM(R))$. Our main result implies that there is a group isomorphism $\alpha : Out(RFM(R)) \to Pic(R)$ for which the diagram

$$\begin{array}{ccc} Out(M_n(R)) & \xrightarrow{\alpha_n} & I_n \\ \cap & & \cap \\ Out(RFM(R)) & \xrightarrow{\alpha} & Pic(R) \end{array}$$

commutes. We note that the group $Out(RFM(R))$ (resp. $Pic(R)$) is in general too large to be the direct union of the subgroups $Out(M_n(R))$ (resp. I_n).

That an isomorphism between the groups $Out(RFM(R))$ and $Pic(R)$ should exist is in part due to a result of Eilenberg (see e.g. [6]), which yields that ${}_RP^{(\mathbf{N})} \cong {}_RR^{(\mathbf{N})}$ for any $P \in Pic(R)$. This suggests that the subgroup $I_{\mathbf{N}}$ of $Pic(R)$ might naturally be defined to be all of $Pic(R)$; from this perspective, the isomorphism

$Out(RFM(R)) \to Pic(R)$ may then be viewed as the 'infinite extension' of the isomorphisms $Out(M_n(R)) \to I_n$ described above.

The ring T is said to have *SBN* (for *single basis number*) provided $T^n \cong T^m$ as left T-modules for all positive integers n, m. It is easy to show that this condition is equivalent to the existence of a left T-module isomorphism $T \cong T \oplus T$.

Lemma 2.1. *Let A be a ring with local units, and let $B = End({}_AA)$. Then A has SBN if and only if B has SBN.*

Proof. Suppose A has SBN; so there are maps $\alpha : A \times A \to A$ and $\beta : A \to A \times A$ such that $\alpha\beta = 1_{A\times A}$ and $\beta\alpha = 1_A$. We can write

$$\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} c & d \end{pmatrix},$$

where $a, b, c, d \in B$, $ac = bd = 1_A$, $ad = 0 = bc$, and $ac + bd = 1_A$. But since these are elements of B as well, then we can view them as right multiplication on B; i.e., we have maps $\rho_\alpha : B \times B \to B$ and $\rho_\beta : B \to B \times B$ such that ρ_α and ρ_β are right multiplications by α and β, respectively. Since the relations above still hold, namely, $\alpha\beta = 1$ and $\beta\alpha = 1$, we see that B has SBN.

Conversely, suppose there are maps $\alpha : B \times B \to B$ and $\beta : B \to B \times B$ such that $\alpha\beta = 1_{B\times B}$ and $\beta\alpha = 1_B$. We can write

$$\alpha = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} c & d \end{pmatrix},$$

where $ac = bd = 1_B$, $ad = 0 = bc$, and $ac+bd = 1_B$. Since B is a ring with identity, the elements a, b, c, d belong to B, which means they are endomorphisms of A; i.e., $a, b, c, d \in End({}_AA)$. It follows that A has SBN. □

Lemma 2.2. *Let R be a unital ring. Let $A = FM(R)$, and let $B = RFM(R)$.*
(1) $B = End({}_AA)$ as right multiplications.
(2) Both A and B have SBN.

Proof. (1) As A is a right ideal of B, right multiplication by an element of $RFM(R)$ is an element of $End({}_AA)$. Now for any positive integer i and $f \in End({}_AA)$ we have $(e_{ii})f = e_{ii}(e_{ii})f$. In particular $(e_{ii})f$ has all rows equal to zero except the i^{th}. Define the matrix M whose i^{th} row is the i^{th} row of $(e_{ii})f$. So in fact $M = \sum_{i\in\mathbf{N}}(e_{ii})f = \sum_{i\in\mathbf{N}} e_{ii} \cdot (e_{ii})f$. Note that $M \in RFM(R)$. But there exists $\{i_1, ..., i_n\} \subseteq \mathbf{N}$ such that $a = a\sum_{k=1}^n e_{i_k i_k}$. Then $(a)f = (a\sum_{k=1}^n e_{i_{ki} i_k})f = a(\sum_{k=1}^n e_{i_{ki} i_k})f = aM$.

(2) The ring $RFM(R)$ is well-known to have SBN (see e.g. [2, Exercise 8.14]). Now apply part (1) with Lemma 2.1. □

Lemma 2.3. *Let T be a ring with SBN.*
(1) For any $P \in Pic(T)$, $P \oplus P \cong P$ as left T-modules.
(2) For any element $P \in J \subseteq Pic(T)$ we have ${}_TP \cong {}_TT$. In particular, for any unital SBN ring R, ${}_RP \cong {}_RR$ for any $P \in Pic(R)$.

Proof. (1) Let $P \in Pic(T)$. We claim that $P \oplus P \cong P$ as left T-modules. Let Q denote P^{-1} in $Pic(T)$. Then as left T-modules we have

$$Q \otimes (P \oplus P) \cong (Q \otimes P) \oplus (Q \otimes P) \cong T \oplus T \cong T \cong Q \otimes P,$$

which with the autoequivalence property of Q yields $P \oplus P \cong P$, as required. We note that this in turn gives $P^m \cong P$ for any integer m.

(2) As $P \in J$ we have that P is a direct summand of T^n for some integer n; this means that P is a direct summand of T (since T has SBN); say $T = P \oplus V$. Similarly, T is a direct summand of P^m for some integer m. But P^m is isomorphic to P by the previous paragraph, so T is a direct summand of P; say $P = T \oplus U$. So we have left T-module isomorphisms

$$\begin{aligned} P = T \oplus U &\cong T \oplus T \oplus U \cong T \oplus P \cong P \oplus V \oplus P \\ &\cong P \oplus P \oplus V \cong P \oplus V \cong T. \end{aligned}$$

The last statement follows since for unital rings R we always have $J = Pic(R)$. ☐

We note that we do not know whether part (1) of the previous lemma is valid for an arbitrary progenerator ${}_TP$ of T.

Corollary 2.4. *$Pic(RFM(R))$ is outer induced for any unital ring R. In particular, there is an isomorphism of groups $\tau : Out(RFM(R)) \to Pic(RFM(R))$.*

Proof. Let B denote $RFM(R)$. By Lemmas 2.2(2) and 2.3 we have that every element P of $Pic(B)$ has ${}_BP \cong {}_BB$. But as B is unital we may apply [8, Theorem 55.12] to conclude that every element of $Pic(B)$ is of the form B_ϕ for some $\phi \in Aut(B)$. ☐

Proposition 2.5. *Let R be a unital ring. Then $Pic(R) \cong Pic(FM(R))$. This isomorphism can be described explicitly as follows. Let A denote $FM(R)$, and let e denote $e(1,1)$, the matrix idempotent with 1 in the (1,1)-entry and zeros elsewhere. Then the group isomorphism from $Pic(R)$ to $Pic(A)$ is given by $X \mapsto Ae \otimes X \otimes eA$. Moreover, $Ae \otimes X \otimes eA \cong FM(X)$, the countably square matrices having at most finitely many nonzero entries, whose entries are elements of X.*

Proof. By definition, for any ring T, $Pic(T)$ denotes the group of autoequivalences of the category T-Mod. But R-Mod and $FM(R)$-Mod $=$ A-Mod are equivalent categories by [3, p.14], hence have isomorphic Picard groups. Since $eAe \cong R$ and $AeA = A$, the map $X \mapsto Ae \otimes X \otimes eA$ is an isomorphism of Picard groups.

It remains to prove that $Ae \otimes X \otimes eA \cong FM(X)$. Define $\psi : Ae \otimes X \otimes eA \to FM(X)$ via $(z)\psi = [z]$, where the (i,j)-entry of $[z]$ is given by $e(i,i) \cdot z \cdot e(j,j)$. Here we identify $e(i,i)Ae(1,1)$ with R, and, similarly, we identify $e(1,1)Ae(j,j)$ with R.

It is straightforward to show that ψ is a linear bijection; we need only show that ψ is an $A - A$ bimodule map. To do this, it suffices to show $(e(k,l)ze(u,v))\psi = e(k,l)(z)\psi e(u,v)$. But $e(k,l)ze(u,v)$ consists of zeros off row k and off column v; in fact, $e(k,l)ze(u,v)$ consists of the (l,u)-entry from z in the (k,v)-position and zeros elsewhere. Similarly, $e(k,l)[z]e(u,v)$ has zero entries except in the (k,v)-position, which contains the (l,u)-entry of z. Thus, $[e(k,l)ze(u,v)] = e(k,l)[z]e(u,v)$, and so ψ is a bimodule isomorphism. ☐

A key ingredient in the proof of our main result is the fact, noted by Camillo in [6], that automorphisms of $B = RFM(R)$ behave nicely with respect to the subring $A = FM(R)$. We show now how Camillo's idea can be tailored to the situation at hand.

Lemma 2.6. *Let A be a ring with local units, and let $B = End({}_AA)$.*

(1) Let M be a left A-module. Then M is finitely generated (in the categorical sense) if and only if there exist $x_1, ..., x_n \in M$ with $M = Ax_1 + \cdots + Ax_n$.

(2) Let $b \in B$. Then ${}_AAb$ is contained in a finitely generated submodule of ${}_AA$ if and only if $b \in BA$.

(3) Suppose that ${}_AAe^\tau$ is finitely generated for each $\tau \in Aut(B)$ and each $e \in E$. Then $B_\sigma \in H$ for each $\sigma \in Aut(B)$.

Proof. (1) The categorical definition of finitely generated is: for any collection $\{M_\alpha\}_{\alpha \in I}$ and for any surjection $f : \bigoplus_{\alpha \in I} M_\alpha \to M$ there is a finite subset $J = \{\alpha_1, ..., \alpha_n\}$ of I such that $f \mid_{\bigoplus_{\beta \in J} M_\beta}$ is surjective. For any *unitary* module M, the map $g : \bigoplus_{x \in M} A \to M$ given by right multiplication by x in the x-th coordinate is a surjection. The result is now clear.

(2) ($\Rightarrow$) By part (1) we have $Ab \subseteq Ax_1 + \cdots + Ax_n$, where $x_1, ..., x_n \in A$. Then there exists $e \in A$ with $x_i e = x_i$ for all i. So for each $ab \in Ab$ we have $abe = ab$. But this yields that $be = b$, since viewed as functions on A these functions are equal for all $a \in A$. Thus $b \in BA$. ($\Leftarrow$) We note that $b \in BA$ if and only if $b = be$ for some $e \in E$; so upon writing $b = be$ we get $Ab = Abe \subseteq Ae$; since $e \in A$ we have that ${}_AAe$ is finitely generated.

(3) By Corollary 1.17 we need only show that $A \subseteq B \cdot A^\sigma$ and $A \subseteq B \cdot A^{\sigma^{-1}}$; this is equivalent to showing that $A^{\sigma^{-1}} \subseteq BA$ and $A^\sigma \subseteq BA$. But letting $\tau = \sigma$, from part (2) and the hypotheses we get that $e^\sigma \in BA$ for all $e \in E$, so that $a^\sigma = (ae)^\sigma$ (for some $e \in E$) $= a^\sigma e^\sigma \in B \cdot BA \subseteq BA$ for all $a \in A$. The proof that $A^{\sigma^{-1}} \subseteq B \cdot A$ is similar, letting $\tau = \sigma^{-1}$. □

Proposition 2.7 (Camillo [6, p. 188])**.** *Let R be any unital ring, let $A = FM(R)$, and let $B = RFM(R)$. Let E denote the set of local units of A consisting of finite sums of the standard matrix units $\{e_{ii}\}_{i \in \mathbf{N}}$. Then for any $\tau \in Aut(B)$ and $e \in E$ we have that ${}_AAe^\tau$ is finitely generated. In particular, $B_\sigma \in H$ for every $\sigma \in Aut(B)$.*

Proof. Since any ${}_AAe^\tau$ is a finite direct sum of left A-modules of the form ${}_AAe^\tau_{ii}$, we need only show that each ${}_AAe^\tau_{ii}$ is finitely generated. But ${}_AAe^\tau_{11} \cong {}_AAe^\tau_{ii}$ for all $i \geq 1$; the isomorphism is given by right multiplication by e^τ_{1i}, with right multiplication by e^τ_{i1} as the inverse map. Thus it suffices to show that ${}_AAe^\tau_{11}$ is finitely generated.

For each $a \in A$ we have $ae^\tau_{11} = \sum_{i=1}^\infty ae^\tau_{11}e_{ii}$, since $ae^\tau_{11} \in A$ means that the sum is actually finite. Thus, using the fact that e^τ_{11} is idempotent, we get $ae^\tau_{11} = ae^\tau_{11} \cdot e^\tau_{11} = \sum_{i=1}^\infty ae^\tau_{11}e_{ii}e^\tau_{11}$.

We claim that ${}_AAe^\tau_{11}$ is finitely generated if and only if $ae^\tau_{11}e_{ii}e^\tau_{11} = 0$ for almost all integers i and for all $a \in A$. To see this, suppose Ae^τ_{11} is finitely generated. Then by Lemma 2.6(2) we have that $e^\tau_{11} \in BA$, so there exists a set of integers $\{i_1, ..., i_n\}$ such that $e^\tau_{11} = e^\tau_{11} \sum_{t=1}^n e_{i_t i_t}$. But then $e^\tau_{11}e_{ii} = 0$ for all $i \notin \{i_1, ..., i_n\}$, and the result is clear. Conversely, we use the observation of the previous paragraph to get $ae^\tau_{11} = \sum_{i=1}^\infty ae^\tau_{11}e_{ii}e^\tau_{11}$ for each $a \in A$. So if terms of the form $ae^\tau_{11}e_{ii}e^\tau_{11}$ are zero for almost all i, then this equation says that the finite set $\{e_{jj}e^\tau_{11} \mid j \in \mathbf{N}$ has $ae^\tau_{11}e_{jj}e^\tau_{11}$ nonzero for some $a \in A\}$ generates Ae^τ_{11}.

Now define $h \in End({}_AA) = B = RFM(R)$ by setting, for each $a \in A$,

$$(a)h = \sum_{k=1}^\infty ae^\tau_{11}e_{kk}e^\tau_{1k}.$$

This is well defined, since for each $a \in A$ the element $ae^\tau_{11}e_{kk}$ is nonzero for at most finitely many integers k. It is easy to show that h is left A-linear, so that h is indeed in $End({}_AA)$. In particular, we may apply τ^{-1} to h to obtain the element

$h^{\tau^{-1}}$ of B. Since $e_{11} \in A$ we have $e_{11}h^{\tau^{-1}} \in A$, so in particular $e_{11}h^{\tau^{-1}}e_{ii} = 0$ for almost all i. Now apply τ to both sides to get $e_{11}^{\tau}he_{ii}^{\tau} = 0$ for almost all i; i.e., $ae_{11}^{\tau}he_{ii}^{\tau} = 0$ for almost all i, for all $a \in A$. Using the definition of h, this gives $(\sum_{k=1}^{\infty} ae_{11}^{\tau} \cdot e_{11}^{\tau}e_{kk}e_{1k}^{\tau})e_{ii}^{\tau} = (\sum_{k=1}^{\infty} ae_{11}^{\tau}e_{kk}e_{1k}^{\tau})e_{ii}^{\tau} = 0$ for all $a \in A$, and for almost all i. But $e_{1k}^{\tau}e_{ii}^{\tau} = e_{1i}^{\tau}$ when $k = i$, and is zero otherwise. Using this, all terms but one in the sum become zero, and we get $ae_{11}^{\tau}e_{ii}e_{1i}^{\tau} = 0$ for all $a \in A$ and almost all i. Now multiplying both sides on the right by e_{i1}^{τ} we get $ae_{11}^{\tau}e_{ii}e_{11}^{\tau} = 0$ for all $a \in A$ and almost all i.

Therefore, by the claim established in the previous paragraph we conclude that Ae_{11}^{τ} is finitely generated, which in turn yields the first statement of the proposition.

The second statement now follows immediately from Lemma 2.6(3). □

Proposition 2.8. *Let R be a unital ring.*
(1) $Pic(FM(R)) = J$.
(2) $Pic(RFM(R)) = H$.

Proof. (1) Let A denote $FM(R)$. By Proposition 2.5 it suffices to show that $FM(X) \in J$ for any $X \in Pic(R)$. Since R has an identity, $X \in Div(R)$ and $R \in Div(X)$. Thus, there exists an integer n and split epimorphisms $\alpha : X^n \to R$ and $\beta : R^n \to X$. We can write $\alpha = [\alpha_1, \ldots, \alpha_n]^T$, where $\alpha_i : X \to R$; similarly, we can write $\beta = [\beta_1, \ldots, \beta_n]^T$, where $\beta_i : R \to X$.

We define $\hat{\alpha} : FM(X)^n \to A$ and $\hat{\beta} : A^n \to FM(X)$ as follows. Let $\hat{\alpha_i}$ be the scalar matrix with α_i on the diagonal. This is an infinite matrix. Define $\hat{\beta_i}$ similarly, and set $\hat{\alpha} = [\hat{\alpha_1}, \ldots, \hat{\alpha_n}]^T$ and $\hat{\beta} = [\hat{\beta_1}, \ldots, \hat{\beta_n}]^T$. It is easy to check that these are split A-epimorphisms, which yields $FM(X) \in J$.

(2) Let B denote $RFM(R)$, and let $X \in Pic(B)$. By Corollary 2.4 we have $X \cong B_{\sigma}$ as bimodules for some $\sigma \in Aut(B)$. So it suffices to show that $B_{\sigma} \in H$. But this fact has been established in Proposition 2.7 above. □

Theorem 2.9. *For any unital ring R we have isomorphisms of Picard groups*

$$Pic(R) \cong Pic(FM(R)) \cong Pic(RFM(R)).$$

Proof. The first isomorphism is given in Proposition 2.5. By the previous proposition we have $J = Pic(FM(R))$ and $H = Pic(RFM(R))$. Thus the second isomorphism follows from Theorem 1.14. □

As one consequence of the previous theorem we conclude that for any ring R there is a ring B (namely, $B = RFM(R)$) for which $Pic(B)$ is outer induced, and for which $Pic(R) \cong Pic(B)$. That the groups $Pic(R)$ and $Pic(RFM(R))$ are always isomorphic is perhaps surprising, as the rings R and $RFM(R)$ are rarely Morita equivalent.

Using Theorem 2.9, we can now show that the group of outer automorphisms of $RFM(R)$ which fix R elementwise is abelian when R is commutative. This extends a result of Rosenberg and Zelinsky which first appeared in 1961; see [12].

Corollary 2.10. *Let R be a commutative unital ring. Let $Out_R(RFM(R))$ denote the outer automorphisms of $RFM(R)$ which fix the scalar matrices of $RFM(R)$. Then $Out_R(RFM(R))$ is an abelian group. In fact, $Out_R(RFM(R)) \cong Picent(R)$.*

Proof. We start by noting that since R is commutative, any inner automorphism of $RFM(R)$ fixes the scalar matrices of $RFM(R)$; thus $Out_R(RFM(R))$ is well-defined. For any ring T with center Z we define $Picent(T) = \{X \in Pic(T) \mid zx =$

xz for all $x \in X, z \in Z\}$. If R is commutative then $Picent(R)$ is an abelian group, since the map which takes $x \otimes y \in X \otimes_R Y$ to $y \otimes x \in Y \otimes_R X$ is easily shown to be an isomorphism (using the above definition, and the fact that $Z = R$). Thus it suffices to show that $Out_R(RFM(R)) \cong Picent(R)$.

Let X be an element of $Pic(RFM(R))$; so by Corollary 2.4 we have $X = RFM(R)_\alpha$ for some $\alpha \in Aut(RFM(R))$. We claim that $X \in Picent(RFM(R))$ if and only if α fixes the scalar matrices of $RFM(R)$; i.e., if and only if α generates an element of $Out_R(RFM(R))$ inside $Out(RFM(R))$. To see this, we note that the scalar matrices of $RFM(R)$ form the center of $RFM(R)$, and are isomorphic to R. Now if $X \in Picent(RFM(R))$ and $r \in R$, then for all $x \in X$ we have $x * r = rx$, which gives $xr^\alpha = rx$. But since any ring automorphism preserves centers we get that r^α is in the center of $RFM(R)$, so $xr^\alpha = r^\alpha x$. Thus we get $rx = r^\alpha x$ for all $x \in X$, which gives $(r - r^\alpha)x = 0$ for all $x \in X$, so that $r = r^\alpha$ as X is faithful. The converse statement is proved in a similar way.

Now the isomorphism $\theta : Pic(R) \to Pic(RFM(R))$ described subsequent to Example 2.11 is easily seen to take $Picent(R)$ to $Picent(RFM(R))$. But the above claim yields that $Picent(RFM(R)) \cong Out_R(RFM(R))$, and we are done. □

In fact, Rosenberg and Zelinsky also show that $Out_R(M_n(R))$ has finite exponent dividing n. In contrast, however, an easy example shows that $Out_R(RFM(R))$ can be torsion-free.

Example 2.11. Let R be a Dedekind domain with $Picent(R) \cong \mathbf{Z}$ (the infinite cyclic group); such a domain exists by [7]. Now apply Corollary 2.10 to conclude that $Out_R(RFM(R)) \cong \mathbf{Z}$, hence is torsion-free. □

As yet another consequence of Theorem 2.9 we see that there is an isomorphism of groups $\theta : Pic(R) \to Pic(RFM(R))$, given by $X \mapsto Hom_A(A, FM(X))$. (Here A denotes $FM(R)$.) Furthermore, by Corollary 2.4 there is an isomorphism of groups $\tau : Out(RFM(R)) \to Pic(RFM(R))$. So

$$\alpha = \tau\theta^{-1} : Out(RFM(R)) \to Pic(R)$$

is an isomorphism. For any integer n we have the embedding $\beta : Out(M_n(R)) \to Out(RFM(R))$ induced by applying an element $\phi \in Aut(M_n(R))$ to each of the $n \times n$ blocks of $RFM(R)$. Finally, for any integer n we have the subgroup I_n of $Pic(R)$ described previously. A straightforward series of computations yields that, with α as defined above, the diagram described at the outset of this section indeed commutes. Thus our isomorphism in some sense provides a 'context' for the subgroups I_n of $Pic(R)$.

As mentioned above, $Pic(R)$ is not in general the union of the subgroups I_n; rephrased, $Out(RFM(R))$ need not be the direct limit of the groups $Out(M_n(R))$. For instance,

Example 2.12. Let k be the field of rationals, and let $R = M_2(k) \oplus k$ (ring direct sum). Then R is semiperfect, so that $I_n = I_1$ for all $n \geq 1$, by [2, Theorem 27.11]. In fact, since $Aut(k)$ is trivial it is easy to show that $Out(R)$ is trivial, so that $\bigcup_n I_n = I_1$ is trivial as well. Thus in order to show that $Pic(R) \neq \bigcup_n I_n$ we need only produce $P \in Pic(R)$ having $_RP$ not isomorphic to $_RR$. Such a module is given by viewing R as the subring $\begin{pmatrix} k & k & 0 \\ k & k & 0 \\ 0 & 0 & k \end{pmatrix}$ of $M_3(k)$, and considering

$P = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ k & k & 0 \end{pmatrix}$. Then $P \in Pic(R)$ as $P \otimes P \cong R$, but ${}_RP$ is not isomorphic to ${}_RR$ by a dimension argument.

It would be interesting to classify those rings R for which $Pic(R) = \bigcup_n I_n$.

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Department of Mathematics, University of Colorado, Colorado Springs, Colorado 80933

E-mail address: abrams@math.uccs.edu

E-mail address: haefner@math.uccs.edu